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Spectral decomposition of the linear elastic tensor for monoclinic symmetry

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Abstract

The compliance fourth-rank tensor related to crystalline or other anisotropic media belonging to the monoclinic crystal system is spectrally decomposed for the first time, and its characteristic values and idempotent fourth-rank tensors are established. Further, it is proven that the idempotent tensors resolve the stress and strain second-rank tensors into eigentensors, thus giving rise to a decomposition of the total elastic strain-energy density into non-interacting strain-energy parts. Several examples of representative inorganic crystals of the monoclinic system illustrate the results of the theoretical analysis. It is also proven that the essential parameters required for a coordinate-invariant characterization of the elastic properties of a crystal exhibiting monoclinic symmetry are both the six characteristic values of the compliance tensor and seven dimensionless parameters. These material constants, referred to as the eigenangles, are shown to be accountable for the orientation of the stress and strain eigentensors, when represented in a stress coordinate system. Finally, the restrictions dictated by the classical thermodynamical argument on the elements of the compliance tensor, which are necessary and sufficient for the elastic strain-energy density to be positive definite, are investigated for the monoclinic symmetry.

1. Introduction

Tensors of the fourth rank embodying the elastic or other property of crystalline anisotropic substances were initially expanded (Srinivasan & Nigam, 1969) as a linear combination of independent elementary tensors, corresponding to scalar coefficients, which remain invariant under orthogonal coordinate transformations. Next, the algebra of fourth-rank tensors of the 32 crystal classes was broken down to irreducible subalgebras (Walpole, 1981, 1984), offering insight into the tensor structure and simplifying considerably the calculations of sums, products and inverses between the tensors.

Conversely, the spectral decomposition was proven (Rychlewski, 1984*a*,*b*) to be the simplest possible decomposition of the elastic compliance **S** or stiffness **C** fourth-rank tensors. Additionally, this decomposition was preferable because of its ability to split these tensors into idempotent fourth-rank tensors, which, in turn, defined energy orthogonal stress and strain eigentensors. Moreover, the spectral decomposition did not correspond to the decompositions of both Walpole and Srinivasan & Nigam, except for the trivial cases of isotropic and cubic symmetry.

Nonetheless, the concepts of elastic eigenvalues, as well as those of stress and strain eigentensors, were introduced by Thomson (Lord Kelvin), who called them the '*Six principal elasticities and principal stress and strain-types of an elastic solid*' (Thomson, 1856, 1878). In a time when access to the tensorial formulation of the mathematical theory of elasticity was not feasible, Thomson clearly perceived and established, using an altogether different terminology, the unsurpassed simplicity introduced through the notion of elastic eigenvalues and eigentensors of compliances in the analysis of the structure of the generally anisotropic linearly elastic solid. It is, however, unfortunate that Todhunter & Pearson (1886–93) criticized with a great deal of skepticism the contribution of Thomson in this area. In fact, Lord Kelvin's formulation was entirely neglected for more than a century until Rychlewski recreated the basic ideas of the analysis reported by Lord Kelvin, exhibiting the mathematical structure of an arbitrary linearly elastic anisotropic body.

Despite the fact that Rychlewski confirmed the application of the spectral decomposition principle on the class of symmetric fourth-rank tensors, he did not proceed to determine the eigenvalues and eigentensors of the corresponding tensors. In fact, these were established subsequently (Theocaris & Philippidis, 1989, 1990, 1991) and, combined with a characteristic angle, the eigenangle ω, provided an invariant specification for the elastic features of a transversely isotropic medium. Then, the three-dimensional spectral decomposition was extended to incorporate the two-dimensional plane stress conditions (Theocaris & Sokolis, 1998).

In this paper, a full reduction of the compliance fourth-rank tensor **S** is developed for crystalline media belonging to the monoclinic system, based on the spectral decomposition principle, for the first time. The characteristic values of the compliance tensor **S** are determined and the elementary idempotent fourth-rank tensors are established. These elementary tensors give rise to stress and strain eigentensors, which split the

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elastic potential of the monoclinic medium into distinct elements, designating the absence of a pure dilatational strain-energy component. Further, it is proven that the constitutive parameters, required for an invariant characterization of the elastic properties of a crystal of the monoclinic syngony, are the six distinct eigenvalues of the compliance tensor **S**, in addition to a set of seven dimensionless quantities, referred to as the eigenangles, which are responsible for the orientation and alignment of the stress and strain eigentensors in the six-dimensional stress space. Next, the individual criteria in terms of the elements of the compliance tensor, which are necessary and sufficient for the elastic strain energy to be positive definite, are examined for monoclinic symmetry. Finally, several examples of representative inorganic crystals of the monoclinic system illustrate the results of our theoretical analysis.

2. Linear elasticity of anisotropic media

The generalized anisotropic form of Hooke's law (Hooke, 1678) states that each strain component is directly proportional to each stress component or, in symbolic indicial notation:

$$\boldsymbol{\varepsilon} = \mathbf{S} \cdot \boldsymbol{\sigma} \quad \text{or} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \tag{1}$$

where $i, j, k, l = 1, 2$ or 3 and the coefficients of linearity S_{ijkl} are the coefficients of the compliance fourth-rank tensor **S** expressed in a Cartesian coordinate system. It is further assumed that the deformations are measured from the natural stress-free state and the influence of temperature and other fields is insignificant. In addition, it is presumed that the stress tensor $\boldsymbol{\sigma}$, whose components are σ_{ij}, and the linear strain tensor $\boldsymbol{\varepsilon}$, whose components are ε_{ij}, are symmetric (Sokolnikoff, 1956). As equations (1) stand, there are 81 components of the compliance tensor **S** but, owing to the symmetry of the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ tensors, two important symmetry restrictions are imposed on the compliance tensor **S**, namely:

$$S_{ijkl} = S_{jikl}, \quad S_{ijkl} = S_{ijlk}, \tag{2}$$

which reduce the number of independent components of **S** to 36. Next, another symmetry constraint is imposed on the compliance tensor **S**, based on the thermodynamical argument that no work is produced by an elastic medium in a closed loading cycle. This is the symmetry which necessitates that the components with subscripts *ijkl* and *klij* are equal:

$$S_{ijkl} = S_{klij}. \tag{3}$$

These reciprocal relations further reduce the number of distinct compliance components to 21 in the most general case. In addition, reciprocal relations (3) are of thermodynamic origin, hence they are not dependent upon the actual mechanism of elastic behaviour.

It should be noted that many different notations have been proposed for the stress and strain components at various times (Todhunter & Pearson, 1886–93; Voigt, 1910; Love, 1927; Timoshenko & Goodier, 1951; Southwell, 1941; Cady, 1946; Wooster, 1949; Mason, 1950; Lekhnitskii, 1963; Nye, 1957; Hearmon, 1961). We have used Nye's notation throughout this paper.

The generalized Hooke's law, expressed in (1) utilizing a Cartesian fourth-rank tensor index notation, may be represented in matrix notation as follows:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{1122} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{1133} & S_{2233} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{1123} & 2S_{2223} & 2S_{3323} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1113} & 2S_{2213} & 2S_{3313} & 4S_{2313} & 4S_{1313} & 4S_{1312} \\ 2S_{1112} & 2S_{2212} & 2S_{3312} & 4S_{2312} & 4S_{1312} & 4S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}. \tag{4}$$

Hooke's law is often expressed in its contracted notation, the well known Voigt notation, which is represented in the form

$$\boldsymbol{\varepsilon} = \mathbf{s} \cdot \boldsymbol{\sigma} \quad \text{or} \quad \varepsilon_p = s_{pq}\sigma_q \tag{5}$$

or alternatively in the form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}, \tag{6}$$

where $p, q = 1, 2, \ldots, 6$, utilizing a 6×6 matrix **s**. However, one should be cautious employing the Voigt notation, since this is not a tensorial notation, that is the components s_{pq} do not form the components of a tensor as do the S_{ijkl}, which constitute the components of a Cartesian fourth-rank tensor in three dimensions. Nevertheless, the Voigt notation is important because it is almost invariably used in experimental work of elasticity and has become the standard in anisotropic elasticity (Voigt, 1910; Nye, 1957; Hearmon, 1961). Then, the equivalence between the components of the compliance fourth-rank tensor **S** and the components of the 6×6 matrix **s** of the Voigt notation is shown to be

$$S_{ijkl} = s_{pq} \quad \text{for } p, q = 1, 2 \text{ or } 3 \tag{7a}$$

$$2S_{ijkl} = s_{pq} \quad \text{for } p = 1, 2 \text{ or } 3 \text{ and } q = 4, 5 \text{ or } 6 \tag{7b}$$

$$4S_{ijkl} = s_{pq} \quad \text{for } p, q = 4, 5 \text{ or } 6, \tag{7c}$$

in which the following contraction rule is applied for replacing a pair of indices by a single contracted index: $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, $(23, 32) \rightarrow 4$, $(13, 31) \rightarrow 5$, $(12, 21) \rightarrow 6$.

Furthermore, the full tensor suffixes of the stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\varepsilon}$ are contracted according to the scheme:

$$\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3,$$
$$\sigma_{23} = \sigma_4,\ \sigma_{13} = \sigma_5,\ \sigma_{12} = \sigma_6 \tag{8a}$$
$$\varepsilon_{11} = \varepsilon_1,\ \varepsilon_{22} = \varepsilon_2,\ \varepsilon_{33} = \varepsilon_3,$$
$$2\varepsilon_{23} = \varepsilon_4,\ 2\varepsilon_{13} = \varepsilon_5,\ 2\varepsilon_{12} = \varepsilon_6. \tag{8b}$$

The occurrence of the factor 2 in the equations relating to the shear strains in (8*b*) should be particularly noted and the shear strains ε_{ij}, $i, j = 1, 2$ or 3, $i \neq j$, carefully distinguished from the contracted shear strains ε_p, $p = 4, 5$ or 6, which do not form the components of a tensor, as do the ε_{ij}.

3. Spectral decomposition of the monoclinic compliance fourth-rank tensor

In this paper, our attention is restricted to the monoclinic crystal system, which is characterized by a plane of elastic symmetry. In the following, the compliance fourth-rank tensor **S** of a monoclinic linear elastic solid is decomposed spectrally for the first time. We assume the Cartesian coordinate system, where the stress and strain tensors are referred to, with the 3 axis oriented normal to the plane of elastic symmetry. The components of the compliance fourth-rank tensor **S**, associated with the adopted Cartesian system, with respect to the components of the 6×6 matrix **s** of the Voigt notation, are given by:

$$S_{1111} = s_{11}, \quad S_{2222} = s_{22}, \quad S_{3333} = s_{33}, \tag{9a}$$
$$S_{1122} = S_{2211} = s_{12}, \quad S_{2233} = S_{3322} = s_{23},$$
$$S_{1133} = S_{3311} = s_{13}, \tag{9b}$$
$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = \tfrac{1}{4}s_{44}, \tag{9c}$$
$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = \tfrac{1}{4}s_{55}, \tag{9d}$$
$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \tfrac{1}{4}s_{66}, \tag{9e}$$
$$S_{1323} = S_{1332} = S_{3123} = S_{3132} = S_{2313} = S_{3213} = S_{2331} = S_{3231} = \tfrac{1}{4}s_{45}, \tag{9f}$$
$$S_{1112} = S_{1121} = S_{1211} = S_{2111} = \tfrac{1}{2}s_{16}, \tag{9g}$$
$$S_{2212} = S_{2221} = S_{1222} = S_{2122} = \tfrac{1}{2}s_{26}, \tag{9h}$$
$$S_{3312} = S_{3321} = S_{1233} = S_{2133} = \tfrac{1}{2}s_{36}, \tag{9i}$$

and all the remaining S_{ijkl} components are zero.

The eigenvalues of the square matrix of rank six associated with the tensor **S** were determined by solving its characteristic equation:

$$\det\begin{bmatrix} s_{11}-\lambda & s_{12} & s_{13} & 0 & 0 & s_{16}/2^{1/2} \\ s_{12} & s_{22}-\lambda & s_{23} & 0 & 0 & s_{26}/2^{1/2} \\ s_{13} & s_{23} & s_{33}-\lambda & 0 & 0 & s_{36}/2^{1/2} \\ 0 & 0 & 0 & s_{44}/2-\lambda & s_{45}/2 & 0 \\ 0 & 0 & 0 & s_{45}/2 & s_{55}/2-\lambda & 0 \\ s_{16}/2^{1/2} & s_{26}/2^{1/2} & s_{36}/2^{1/2} & 0 & 0 & s_{66}/2-\lambda \end{bmatrix} = 0 \tag{10}$$

introducing factors $1/2^{1/2}$ and $1/2$ in order to operate with the 6×6 matrices associated with the compliance **S** and stiffness **C** fourth-rank tensors using tensorial rules. In fact, with this modification, the components of the associated 6×6 matrices form the components of a Cartesian second-rank tensor in six dimensions. Equation (10) is then equivalent to

$$(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D) \times \left[\lambda^2 - \lambda\left(\frac{s_{44}+s_{55}}{2}\right) + \left(\frac{s_{44}s_{55}-s_{45}^2}{4}\right)\right] = 0 \tag{11}$$

with

$$A = -s_{11} - s_{22} - s_{33} - \frac{s_{66}}{2} \tag{12a}$$
$$B = (s_{11}s_{22} + s_{11}s_{33} + s_{22}s_{33}) + (s_{11} + s_{22} + s_{33})\frac{s_{66}}{2} - \left(s_{12}^2 + s_{13}^2 + s_{23}^2 + \frac{s_{16}^2}{2} + \frac{s_{26}^2}{2} + \frac{s_{36}^2}{2}\right) \tag{12b}$$
$$C = s_{11}\left(s_{23}^2 + \frac{s_{26}^2}{2} + \frac{s_{36}^2}{2}\right) + s_{22}\left(s_{13}^2 + \frac{s_{16}^2}{2} + \frac{s_{36}^2}{2}\right) + s_{33}\left(s_{12}^2 + \frac{s_{16}^2}{2} + \frac{s_{26}^2}{2}\right) + \frac{s_{66}}{2}(s_{12}^2 + s_{13}^2 + s_{23}^2 - s_{11}s_{12} - s_{11}s_{33} - s_{22}s_{33}) - (s_{11}s_{22}s_{33} + s_{23}s_{26}s_{36} + s_{13}s_{16}s_{36} + s_{12}s_{16}s_{26} + 2s_{12}s_{13}s_{23}) \tag{12c}$$
$$D = s_{16}s_{26}(s_{33}s_{12} - s_{13}s_{23}) + s_{26}s_{36}(s_{11}s_{23} - s_{12}s_{13}) + s_{16}s_{36}(s_{22}s_{13} - s_{12}s_{23}) + \frac{s_{66}}{2}(s_{11}s_{22}s_{33} + 2s_{12}s_{13}s_{23} - s_{11}s_{23}^2 - s_{22}s_{13}^2 - s_{33}s_{12}^2) + \frac{s_{16}^2}{2}(s_{23}^2 - s_{22}s_{33}) + \frac{s_{26}^2}{2}(s_{13}^2 - s_{11}s_{33}) + \frac{s_{36}^2}{2}(s_{12}^2 - s_{11}s_{22}). \tag{12d}$$

The polynomial inside the first parentheses of relation (11) is a quartic. Therefore, by substituting $\lambda = y - A/4$, the quartic polynomial is transformed to its reduced form:

$$y^4 + Py^2 + Qy + R = 0, \tag{13}$$

where

$$P = B - \frac{3A^2}{8} \tag{14a}$$
$$Q = \frac{A^3}{8} - \frac{AB}{2} + C \tag{14b}$$
$$R = -3\left(\frac{A}{4}\right)^4 + \frac{A^2B}{16} - \frac{AC}{4} + D. \tag{14c}$$

Relation (13) may be expressed alternatively as follows:

$$\left(y^2 + \frac{z}{2}\right)^2 - \left[(z-P)y^2 - Qy + \left(\frac{z^2}{4} - R\right)\right] = 0. \tag{15}$$

The second term of (15) is a quadratic function of y, whose discriminant Δ is

$$\Delta = z^3 - Pz^2 - 4Rz + 4PR - Q^2. \tag{16}$$

We choose the discriminant equal to zero ($\Delta = 0$), thus allowing the quadratic function to become a perfect square:

$$z^3 - Pz^2 - 4Rz + 4PR - Q^2 = 0. \tag{17}$$

This is a cubic equation, which has to be transformed to its reduced form in order to be solved. Substituting $\mu = z - P/3$ gives the cubic equation in the form

$$\mu^3 + P'\mu + Q' = 0 \tag{18}$$

with

$$P' = -4R - \frac{P^2}{3} \tag{19a}$$

$$Q' = -\frac{2P^3}{27} + \frac{8PR}{3} - Q^2. \tag{19b}$$

Further, with $\mu = q + k/q$, (18) may be recast as

$$q^6 + Q'q^3 + k^3 = 0, \tag{20}$$

in which

$$k = \frac{P'}{3}. \tag{21}$$

Moreover, with $u = q^3$, (20) becomes a quadratic equation, which is readily solved:

$$u^2 + Qu + k^3 = 0. \tag{22}$$

Thus, the three solutions z_m, $m = 1, 2, 3$, of the cubic polynomial of relation (17) were determined to be

$$z_1 = -\left[-\frac{Q'}{2} + \left(\frac{Q'^2}{4} - k^3\right)^{1/2}\right]^{1/3} \frac{[1 + i(3)^{1/2}]}{2} + \frac{k[-1 + i(3)^{1/2}]}{2[-Q'/2 + (Q'^2/4 - k^3)^{1/2}]^{1/3}} + \frac{P}{3} \tag{23a}$$

$$z_2 = \left[-\frac{Q'}{2} + \left(\frac{Q'^2}{4} - k^3\right)^{1/2}\right]^{1/3} \frac{[1 + i(3)^{1/2}]}{2} + \frac{k[-1 - i(3)^{1/2}]}{2[-Q'/2 + (Q'^2/4 - k^3)^{1/2}]^{1/3}} + \frac{P}{3} \tag{23b}$$

$$z_3 = \left[-\frac{Q'}{2} + \left(\frac{Q'^2}{4} - k^3\right)^{1/2}\right]^{1/3} + \frac{k}{[-Q'/2 + (Q'^2/4 - k^3)^{1/2}]^{1/3}} + \frac{P}{3}. \tag{23c}$$

Substitution in relation (15) for $z = z_m$, $m = 1, 2, 3$, gives the quartic equation in the form

$$\left(y^2 + \frac{z_m}{2}\right)^2 - \left[y - \frac{Q}{2(z_m - P)}\right]^2 = 0. \tag{24}$$

Relation (24) leads to two quadratic equations, which are readily solved:

$$y^2 + y \pm \left[\frac{z_m}{2} - \frac{Q}{2(z_m - P)}\right] = 0. \tag{25}$$

The eigenvalues λ_m, $m = 1, \ldots, 6$, of the associated square matrix of rank six to tensor $\mathbf{S}$ defined by (9) were, thus, evaluated to be:

$$\lambda_1 = -\frac{(z_m - P)^{1/2}}{2} + \frac{1}{2}\left[-(z_m + P) + \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4} \tag{26a}$$

$$\lambda_2 = -\frac{(z_m - P)^{1/2}}{2} - \frac{1}{2}\left[-(z_m + P) + \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4} \tag{26b}$$

$$\lambda_3 = \frac{(z_m - P)^{1/2}}{2} + \frac{1}{2}\left[-(z_m + P) - \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4} \tag{26c}$$

$$\lambda_4 = \frac{(z_m - P)^{1/2}}{2} - \frac{1}{2}\left[-(z_m + P) - \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4} \tag{26d}$$

$$\lambda_5 = \frac{s_{44} + s_{55}}{4} + \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2\right]^{1/2} \tag{26e}$$

$$\lambda_5 = \frac{s_{44} + s_{55}}{4} - \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2\right]^{1/2}. \tag{26f}$$

The characteristic values λ_m, $m = 1, \ldots, 6$, defined by relations (26), constitute the roots of the minimum polynomial of the compliance tensor $\mathbf{S}$, which in factorized form may be written as

$$(\mathbf{S} - \lambda_1\mathbf{I}) \ldots (\mathbf{S} - \lambda_6\mathbf{I}) = 0, \tag{27}$$

where $\mathbf{I}$ is the unit element of the symmetric fourth-rank tensor $\mathbf{M}$ space, which in symbolic notation is represented as $\mathbf{M} \equiv \mathrm{sym}(\mathbf{L} \otimes \mathbf{L})$. Furthermore, we refer to a symmetric fourth-rank tensor if this is described by a symmetric 6×6 matrix of the form given in (10), that is, if this tensor satisfies (2) and (3).

The corresponding six idempotent fourth-rank tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, of the spectral decomposition of $\mathbf{S}$ were obtained as:

$$\mathbf{E}_m = \frac{(\mathbf{S} - \lambda_1\mathbf{I}) \ldots (\mathbf{S} - \lambda_{m-1}\mathbf{I})(\mathbf{S} - \lambda_{m+1}\mathbf{I}) \ldots (\mathbf{S} - \lambda_6\mathbf{I})}{(\lambda_m - \lambda_1) \ldots (\lambda_m - \lambda_{m-1})(\lambda_m - \lambda_{m+1}) \ldots (\lambda_m - \lambda_6)}. \tag{28}$$

Tensors $\mathbf{E}_m$ were, thus, evaluated to be:

$$\mathbf{E}_1 = E^1_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij} g_{kl} \tag{29a}$$

$$\mathbf{E}_2 = E^2_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij} r_{kl} \tag{29b}$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij} h_{kl} \tag{29c}$$

$$\mathbf{E}_4 = E^4_{ijkl} = \mathbf{s} \otimes \mathbf{s} = s_{ij} s_{kl} \tag{29d}$$

$$\mathbf{E}_5 = E^5_{ijkl} = \mathbf{t} \otimes \mathbf{t} = t_{ij} t_{kl} \tag{29e}$$

$$\mathbf{E}_6 = E^6_{ijkl} = \mathbf{q} \otimes \mathbf{q} = q_{ij} q_{kl}, \tag{29f}$$

with $\mathbf{g}, \mathbf{r}, \mathbf{h}, \mathbf{s}, \mathbf{t}, \mathbf{q} \in \mathbf{L}$, where $\mathbf{L}$ represents the second-rank symmetric tensor space over $\mathbf{R}^3$, which, together with the ordinary definition of the scalar product, constitutes a 6D Euclidean space. In symbolic notation, the tensor space $\mathbf{L}$ is expressed by $\mathbf{L} \equiv \mathrm{sym}(\mathbf{R}^3 \otimes \mathbf{R}^3)$.

The second-rank symmetric tensors $\mathbf{g}, \mathbf{r}, \mathbf{h}, \mathbf{s}, \mathbf{t}$ and $\mathbf{q}$, appearing in (29) for the expressions of the idempotent tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, are defined as follows:

$$\mathbf{g} = g_3 \mathbf{a} + g_2 \mathbf{b} + g_1 \mathbf{c} + g_6 \mathbf{d} \tag{30a}$$

$$\mathbf{r} = r_3 \mathbf{a} + r_2 \mathbf{b} + r_1 \mathbf{c} + r_6 \mathbf{d} \tag{30b}$$

$$\mathbf{h} = h_3 \mathbf{a} + h_2 \mathbf{b} + h_1 \mathbf{c} + h_6 \mathbf{d} \tag{30c}$$

$$\mathbf{s} = -\sin\rho \mathbf{a} - \sin\nu \cos\rho \mathbf{b} - \sin\mu \cos\rho \cos\nu \mathbf{c} + \cos\rho \cos\nu \cos\mu \mathbf{d} \tag{30d}$$

$$\mathbf{t} = \cos\psi \mathbf{f} + \sin\psi \mathbf{e} \tag{30e}$$

$$\mathbf{q} = -\sin\psi \mathbf{f} + \cos\psi \mathbf{e}, \tag{30f}$$

in which

$$g_1 = \sin\theta \sin\varphi \cos\mu - \sin\mu(-\sin\theta \cos\varphi \sin\nu + \cos\theta \sin\rho \cos\nu) \tag{31a}$$

$$g_2 = -\sin\theta \cos\varphi \cos\nu - \cos\theta \sin\rho \sin\nu \tag{31b}$$

$$g_3 = \cos\theta \cos\rho \tag{31c}$$

$$g_6 = \sin\theta \sin\varphi \sin\mu + \cos\mu(-\sin\theta \cos\varphi \sin\nu + \cos\theta \sin\rho \cos\nu) \tag{31d}$$

$$r_1 = (-\sin\omega \cos\varphi - \cos\theta \sin\varphi \cos\omega) \cos\mu - \sin\mu[(-\sin\omega \sin\varphi + \cos\theta \cos\varphi \cos\omega) \sin\nu + \cos\omega \sin\theta \sin\rho \cos\nu] \tag{31e}$$

$$r_2 = (-\sin\omega \sin\varphi + \cos\theta \cos\varphi \cos\omega) \cos\nu - \cos\omega \sin\theta \sin\rho \sin\nu \tag{31f}$$

$$r_3 = \cos\omega \sin\theta \cos\rho \tag{31g}$$

$$r_6 = (-\sin\omega \cos\varphi - \cos\theta \sin\varphi \cos\omega) \sin\mu + \cos\mu[(-\sin\omega \sin\varphi + \cos\theta \cos\varphi \cos\omega) \sin\nu + \cos\omega \sin\theta \sin\rho \cos\nu] \tag{31h}$$

$$h_3 = (\cos\omega \cos\varphi - \cos\theta \sin\varphi \sin\omega) \cos\mu - \sin\mu[(\cos\omega \sin\varphi + \cos\theta \cos\varphi \sin\omega) \sin\nu + \sin\omega \sin\theta \sin\rho \cos\nu] \tag{31i}$$

$$h_2 = (\cos\omega \sin\varphi - \cos\theta \sin\varphi \sin\omega) \cos\nu - \sin\omega \sin\theta \sin\rho \sin\nu \tag{31j}$$

$$h_1 = \sin\omega \sin\theta \cos\rho \tag{31k}$$

$$h_6 = (\cos\omega \cos\varphi - \cos\theta \sin\varphi \sin\omega) \sin\mu + \cos\mu[(\cos\omega \sin\varphi + \cos\theta \cos\varphi \sin\omega) \sin\nu + \sin\omega \sin\theta \sin\rho \cos\nu]. \tag{31l}$$

Furthermore, the second-rank symmetric tensors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ and $\mathbf{f}$ emerging in relations (30), in the expressions for the second-rank symmetric tensors $\mathbf{g}, \mathbf{r}, \mathbf{h}, \mathbf{s}, \mathbf{t}$ and $\mathbf{q}$ are defined as follows:

$$\mathbf{a} = \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{b} = \mathbf{l} \otimes \mathbf{l}, \quad \mathbf{c} = \mathbf{m} \otimes \mathbf{m} \tag{32a}$$

$$\mathbf{d} = \frac{1}{2^{1/2}}(\mathbf{l} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{l}) \tag{32b}$$

$$\mathbf{e} = \frac{1}{2^{1/2}}(\mathbf{k} \otimes \mathbf{l} + \mathbf{l} \otimes \mathbf{k}) \tag{32c}$$

$$\mathbf{f} = \frac{1}{2^{1/2}}(\mathbf{k} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{k}) \tag{32d}$$

with $\mathbf{k}, \mathbf{l}$ and $\mathbf{m}$ being the unit vectors of $\mathbf{R}^3$, associated with the 3, 2 and 1 directions of the Cartesian coordinate system.

Further, according to (9), it is easily noted that the components of tensor $\mathbf{S}$ are both symmetrical and real, thus, it follows that tensor $\mathbf{S}$ is self-adjoint or hermitian. Hence, the proof that all the eigenvalues λ_m and idempotent fourth-rank tensors $\mathbf{E}_m, m = 1, \ldots, 6$, of the spectral decomposition of $\mathbf{S}$ are real is obtained at once, based on the hermitian nature of the compliance fourth-rank tensor $\mathbf{S}$.

In addition, the seven angles $\rho, \nu, \mu, \psi, \theta, \omega$ and φ, appearing in relations (31), are called eigenangles and are defined as follows:

$$\tan\rho = \frac{Q_4}{(Z_4^2 + W_4^2 + 1)^{1/2}} \quad \tan\nu = \frac{W_4}{(Z_4^2 + 1)^{1/2}}, \quad \tan\mu = Z_4 \tag{33a}$$

$$\cos 2\psi = \left[\frac{(s_{44} - s_{55})}{2}\right]\left[\left(\frac{s_{44} - s_{55}}{2}\right)^2 + s_{45}^2\right]^{-1/2} \tag{33b}$$

$$\tan\theta = \left(\frac{Z_4^2 + W_4^2 + 1}{Z_4^2 + W_4^2 + Q_4^2 + 1} - \frac{Q_1^2}{Z_1^2 + W_1^2 + Q_1^2 + 1}\right)^{1/2} \times \left[\frac{Q_1}{(Z_1^2 + W_1^2 + Q_1^2 + 1)^{1/2}}\right]^{-1} \tag{33c}$$

$$\tan\omega = \left(\frac{Z_4^2 + W_4^2 + 1}{Z_4^2 + W_4^2 + Q_4^2 + 1} - \frac{Q_1^2}{Z_1^2 + W_1^2 + Q_1^2 + 1} - \frac{Q_2^2}{Z_2^2 + W_2^2 + Q_2^2 + 1}\right)^{1/2} \times \left[\frac{Q_2}{(Z_2^2 + W_2^2 + Q_2^2 + 1)^{1/2}}\right]^{-1} \tag{33d}$$

$$\tan\varphi = \left\{\left(\frac{Z_4^2+W_4^2+1}{Z_4^2+W_4^2+Q_4^2+1}-\frac{Q_1^2}{Z_1^2+W_1^2+Q_1^2+1}\right)\times\left[\frac{(Z_4^2+W_4^2+1)^2}{(Z_4^2+W_4^2+Q_4^2+1)^2}\right]^{-1}(Z_4^2+1)-\frac{Q_1^2Q_4^2W_4^2}{(Z_1^2+W_1^2+Q_1^2+1)(Z_4^2+W_4^2+1)^2}\right\}^{1/2}\times\left[\frac{W_2}{(Z_2^2+W_2^2+Q_2^2+1)^{1/2}}+\frac{Q_1Q_4W_4}{(Z_1^2+W_1^2+Q_1^2+1)^{1/2}(Z_4^2+W_4^2+1)}\right]^{-1}, \tag{33e}$$

in which

$$Q_i = \left[\frac{F_i - C_i^2/2A_i}{2^{-1/2}(E_i - B_iC_i/A_i)}\right],$$

$$W_i = \frac{B_i}{A_i}\left[\frac{F_i - C_i^2/2A_i}{2^{-1/2}(E_i - B_iC_i/A_i)}\right] - \frac{C_i}{2^{1/2}A_i} \tag{34a}$$

$$Z_i = -\frac{s_{12}}{(s_{11}-\lambda_i)}\left\{\frac{B_i}{A_i}\left[\frac{F_i - C_i^2/2A_i}{2^{-1/2}(E_i - B_iC_i/A_i)}\right] - \frac{C_i}{2^{1/2}A_i}\right\} + \frac{s_{13}}{(s_{11}-\lambda_i)}\left[\frac{F_i - C_i^2/2A_i}{2^{-1/2}(E_i - B_iC_i/A_i)}\right] - \frac{s_{16}}{2^{1/2}(s_{11}-\lambda_i)}, \tag{34b}$$

where

$$A_i = \left[(s_{22}-\lambda_i) - \frac{s_{12}^2}{(s_{11}-\lambda_i)}\right],$$

$$B_i = \left[s_{23} - \frac{s_{12}s_{13}}{(s_{11}-\lambda_i)}\right] \tag{35a}$$

$$C_i = \left[s_{26} - \frac{s_{12}s_{16}}{(s_{11}-\lambda_i)}\right],$$

$$D_i = \left[(s_{33}-\lambda_i) - \frac{s_{13}^2}{(s_{11}-\lambda_i)}\right] \tag{35b}$$

$$E_i = \left[s_{36} - \frac{s_{13}s_{16}}{(s_{11}-\lambda_i)}\right],$$

$$F_i = \left[\left(\frac{s_{66}}{2}-\lambda_i\right) - \frac{s_{16}^2}{2(s_{11}-\lambda_i)}\right] \tag{35c}$$

and the subscript i acquires the values 1, 2, 3 or 4.

For the eigenvalues λ_m, $m = 1, \ldots, 6$, given by relations (26), and the corresponding idempotent fourth-rank tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, expressed by relations (29), the compliance fourth-rank tensor $\mathbf{S}$ is spectrally decomposed. It is, hence, given the following expansion:

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \ldots + \lambda_6\mathbf{E}_6. \tag{36}$$

Therefore, the six eigenvalues λ_m, $m = 1, \ldots, 6$, together with the eigenangles $\rho, \nu, \mu, \psi, \theta, \omega$ and φ constitute the 13 coordinate-invariant parameters necessary for the characterization of the elastic properties of crystals belonging to the monoclinic syngony.

Furthermore, the elementary idempotent tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, decompose the unit element $\mathbf{I}$ of the fourth-rank symmetric tensor space $\mathbf{M}$ and satisfy the following set of equations:

$$\mathbf{I} = \mathbf{E}_1 + \ldots + \mathbf{E}_6 \tag{37a}$$

$$\mathbf{E}_m \cdot \mathbf{E}_n = 0, \quad m \neq n \tag{37b}$$

$$\mathbf{E}_m \cdot \mathbf{E}_m = \mathbf{E}_m. \tag{37c}$$

In fact, the idempotent fourth-rank tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, provide an orthogonal expansion of the space $\mathbf{M}$ of symmetric fourth-rank tensors into orthogonal subspaces $\mathbf{M}_m$ as follows:

$$\mathbf{M} = \mathbf{M}_1 \oplus \ldots \oplus \mathbf{M}_6, \quad \mathbf{M}_m \perp \mathbf{M}_n \quad \text{for} \quad m \neq n, \tag{38}$$

where $\mathbf{E}_m$ is the idempotent tensor on $\mathbf{M}_m$ for $m = 1, \ldots, 6$.

4. Energy orthogonal states of stress and strain

The action of the idempotent fourth-rank tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, on the symmetric second-rank tensor space $\mathbf{L}$ leads to a decomposition of the $\mathbf{L}$ space into subspaces $\mathbf{L}_m$ in the following manner:

$$\mathbf{L} = \mathbf{L}_1 \oplus \ldots \oplus \mathbf{L}_6, \quad \mathbf{L}_m \perp \mathbf{L}_n \quad \text{for} \quad m \neq n. \tag{39}$$

Therefore, the stress second-rank eigentensors $\overline{\boldsymbol{\sigma}_m}$ of the compliance fourth-rank tensor $\mathbf{S}$ for the monoclinic symmetry are derived by the orthogonal projection of a second-rank symmetric tensor $\boldsymbol{\sigma}$ on subspaces $\mathbf{L}_m$, produced by the idempotent fourth-rank tensors $\mathbf{E}_m$, as follows:

$$\overline{\boldsymbol{\sigma}_m} = \mathbf{E}_m \cdot \boldsymbol{\sigma}, \quad m = 1, \ldots, 6. \tag{40}$$

Moreover, if the second-rank stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ constitute eigenstates of tensor $\mathbf{S}$, they should satisfy the eigenvalue equation

$$\mathbf{S} \cdot \overline{\boldsymbol{\sigma}_m} = (\lambda_1\mathbf{E}_1 + \ldots + \lambda_6\mathbf{E}_6) \cdot \overline{\boldsymbol{\sigma}_m} = \lambda_m\overline{\boldsymbol{\sigma}_m}, \tag{41}$$

in which index m varies between 1 and 6, and the λ_m values are described in terms of relations (26).

Denoting by $\boldsymbol{\sigma}$ the contracted stress tensor in the form of a 6D vector, which is expressed by

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^T \tag{42}$$

and performing the computations implied by relations (40), it was found that, in contracted notation:

$$\overline{\boldsymbol{\sigma}_1} = (g_1\sigma_1 + g_2\sigma_2 + g_3\sigma_3 + g_6\sigma_6) \times [g_1, g_2, g_3, 0, 0, g_6]^T \quad (43a)$$

$$\overline{\boldsymbol{\sigma}_2} = (r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 + r_6\sigma_6) \times [r_1, r_2, r_3, 0, 0, r_6]^T \quad (43b)$$

$$\overline{\boldsymbol{\sigma}_3} = (h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 + h_6\sigma_6) \times [h_1, h_2, h_3, 0, 0, h_6]^T \quad (43c)$$

$$\overline{\boldsymbol{\sigma}_4} = (-\sin\mu\cos\rho\cos\nu\sigma_1 - \sin\nu\cos\rho\sigma_2 - \sin\rho\sigma_3 + \cos\rho\cos\nu\cos\mu\sigma_6) \times [-\sin\mu\cos\rho\cos\nu, -\sin\nu\cos\rho, -\sin\rho, 0, 0, \cos\rho\cos\nu\cos\mu]^T \quad (43d)$$

$$\overline{\boldsymbol{\sigma}_5} = (\cos\psi\sigma_4 + \sin\psi\sigma_5)[0, 0, 0, \cos\psi, \sin\psi, 0]^T \quad (43e)$$

$$\overline{\boldsymbol{\sigma}_6} = (-\sin\psi\sigma_4 + \cos\psi\sigma_5) \times [0, 0, 0, -\sin\psi, \cos\psi, 0]^T, \quad (43f)$$

where g_i, r_i and h_i, $i = 1, 2, 3, 6$, are defined by relations (31).

Relations (43) assert that the stress eigentensors, corresponding to the spectral decomposition of the compliance tensor **S** for a medium exhibiting monoclinic symmetry, decompose the generic stress tensor $\boldsymbol{\sigma}$ into six elements, namely,

$$\boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}_1} + \ldots + \overline{\boldsymbol{\sigma}_6} \quad (44)$$

with stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_4}$ being a superposition of simple shear with stressing along the 1, 2 and 3 directions of the adopted Cartesian coordinate system, and stress eigentensors $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ constituting simple shear states.

In addition, it is readily observed in relations (43) that the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_4}$ are dependent on the values of eigenangles ρ, ν, μ, θ, ω and φ, expressed by relations (33), in terms of the components of the compliance tensor **S** of the monoclinic body. On the contrary, the remaining two contracted stress eigentensors, namely $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$, are dependent on the value of eigenangle ψ, defined by relation (33*b*).

It is of interest to note that the generalized anisotropic form of Hooke's law, represented by equation (1), may be expressed as follows:

$$\boldsymbol{\varepsilon} = \mathbf{S}\cdot\boldsymbol{\sigma} = (\lambda_1\mathbf{E}_1 + \ldots + \lambda_6\mathbf{E}_6)\cdot\boldsymbol{\sigma} = \lambda_1\overline{\boldsymbol{\sigma}_1} + \ldots + \lambda_6\overline{\boldsymbol{\sigma}_6}, \quad (45)$$

so that the strain second-rank tensor $\boldsymbol{\varepsilon}$ is readily split into six eigentensors $\overline{\boldsymbol{\varepsilon}_m}$:

$$\boldsymbol{\varepsilon} = \overline{\boldsymbol{\varepsilon}_1} + \ldots + \overline{\boldsymbol{\varepsilon}_6}. \quad (46)$$

Therefore, the expression of Hooke's law for crystals belonging to the monoclinic system may be decomposed into six independent laws of proportionality of stress and strain eigentensors in a well defined manner:

$$\overline{\boldsymbol{\varepsilon}_m} = \lambda_m\overline{\boldsymbol{\sigma}_m}, \quad \text{for } m = 1, \ldots, 6. \quad (47)$$

Next, considering the definition of the total elastic strain-energy density, we have that:

$$\begin{aligned} 2T(\boldsymbol{\sigma}) &= \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} \\ &= \boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} \\ &= (\overline{\boldsymbol{\sigma}_1} + \ldots + \overline{\boldsymbol{\sigma}_6})\cdot(\lambda_1\mathbf{E}_1 + \ldots + \lambda_6\mathbf{E}_6) \\ &\quad\cdot(\overline{\boldsymbol{\sigma}_1} + \ldots + \overline{\boldsymbol{\sigma}_6}) \\ &= \lambda_1\overline{\boldsymbol{\sigma}_1}\cdot\overline{\boldsymbol{\sigma}_1} + \ldots + \lambda_6\overline{\boldsymbol{\sigma}_6}\cdot\overline{\boldsymbol{\sigma}_6}. \end{aligned} \quad (48)$$

Relation (48) may be recast as

$$2T(\boldsymbol{\sigma}) = T(\overline{\boldsymbol{\sigma}_1}) + \ldots + T(\overline{\boldsymbol{\sigma}_6}) = \overline{\boldsymbol{\sigma}_1}\cdot\overline{\boldsymbol{\varepsilon}_1} + \ldots + \overline{\boldsymbol{\sigma}_6}\cdot\overline{\boldsymbol{\varepsilon}_6}, \quad (49)$$

that is the elastic potential is decomposed into distinct energy components, each associated with the same stress eigentensor. Denoting by $T(\overline{\boldsymbol{\sigma}_m})$ the following quantity:

$$T(\overline{\boldsymbol{\sigma}_m}) = \lambda_m(\overline{\boldsymbol{\sigma}_m}\cdot\overline{\boldsymbol{\sigma}_m}), \quad m = 1, \ldots, 6, \quad (50)$$

it is noted that any stress eigenstate $\overline{\boldsymbol{\sigma}_m}$ is associated with its own potential $T(\overline{\boldsymbol{\sigma}_m})$, which does not rely on the action of the other $\overline{\boldsymbol{\sigma}_m}$. Then, it is readily noted by inspection of relations (43) that the elastic strain-energy-density components $T(\overline{\boldsymbol{\sigma}_1}), \ldots, T(\overline{\boldsymbol{\sigma}_4})$ are dependent upon the values of the eigenangles ρ, ν, μ, θ, ω and φ, and correspond to both distortional and voluminal alterations of the medium. On the contrary, the last two elastic strain-energy components, namely $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$, are dependent on the value of the eigenangle ψ and are related exclusively to shape distortion of the medium.

5. Geometric representation of stress eigentensors

A direct geometric representation of the $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ contracted stress eigentensors arises if we consider the projections of the stress eigentensors on the shear stress plane (σ_4, σ_5). Then, tensors $\overline{\boldsymbol{\sigma}_1}$ to $\overline{\boldsymbol{\sigma}_4}$ vanish, whereas tensors $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ are represented by two orthogonal unit vectors $\mathbf{e}_5$ and $\mathbf{e}_6$, shown in Fig. 1:

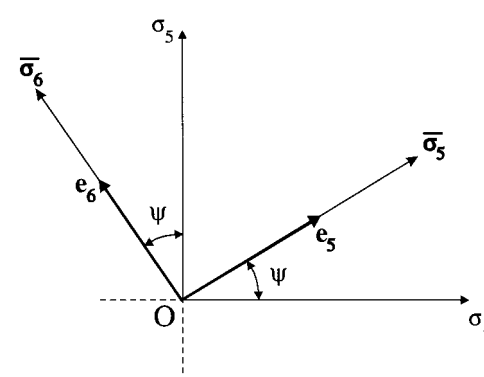

Fig. 1. Geometric representation of the stress eigentensors $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ of the compliance fourth-rank tensor **S** related to monoclinic media on the shear stress plane (σ_4, σ_5).

$$\mathbf{e}_5 = [\cos\psi, \sin\psi]^T, \quad \mathbf{e}_6 = [-\sin\psi, \cos\psi]^T. \tag{51}$$

It is, therefore, noted that vectors $\mathbf{e}_5$ and $\mathbf{e}_6$ subtend angle ψ with respect to axes σ_4 and σ_5 of the shear stress plane (σ_4, σ_5). Thus, an interesting geometric interpretation is offered for eigenangle ψ as the angle responsible for the alignment of stress eigenstates $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ in the shear stress plane (σ_4, σ_5). Besides, by projecting the stress eigentensors on the four-dimensional space system $(\sigma_1, \sigma_2, \sigma_3, \sigma_6)$, tensors $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ disappear, whereas tensors $\overline{\boldsymbol{\sigma}_1}$ to $\overline{\boldsymbol{\sigma}_4}$ are represented by the following orthonormal vectors $\mathbf{e}_m$, $m = 1, \ldots, 4$:

$$\mathbf{e}_1 = [g_1, g_2, g_3, g_6]^T \tag{52a}$$

$$\mathbf{e}_2 = [r_1, r_2, r_3, r_6]^T \tag{52b}$$

$$\mathbf{e}_3 = [h_1, h_2, h_3, h_6]^T \tag{52c}$$

$$\mathbf{e}_4 = [-\sin\mu\cos\rho\cos\nu, -\sin\nu\cos\rho, -\sin\rho, \cos\rho\cos\nu\cos\mu]^T. \tag{52d}$$

In fact, the unit vectors $\mathbf{e}_m$, $m = 1, \ldots, 4$, are the base vectors of a coordinate system obtained by rotating the stress space $(\sigma_1, \sigma_2, \sigma_3, \sigma_6)$ successively through angles ω, θ, φ, ρ, ν and μ, by means of the following transformation matrices $\mathbf{A}_m$, $m = 1, \ldots, 6$:

$$\mathbf{A}_1 = \begin{bmatrix} \cos\omega & \sin\omega & 0 & 0 \\ -\sin\omega & \cos\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{53a}$$

$$\mathbf{A}_3 = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\rho & \sin\rho \\ 0 & 0 & -\sin\rho & \cos\rho \end{bmatrix} \tag{53b}$$

$$\mathbf{A}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\nu & 0 & \sin\nu \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\nu & 0 & \cos\nu \end{bmatrix},$$

$$\mathbf{A}_6 = \begin{bmatrix} \cos\mu & 0 & 0 & \sin\mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\mu & 0 & 0 & \cos\mu \end{bmatrix}. \tag{53c}$$

Then, the complete transformation $\mathbf{A}$ is obtained by considering the product of the six transformation matrices, namely:

$$\mathbf{A} = \prod_{m=1}^{6} \mathbf{A}_m, \tag{54}$$

which is an orthogonal matrix. Therefore, it is concluded that eigenangles ω, θ, φ, ρ, ν and μ determine the orientation of eigentensors $\overline{\boldsymbol{\sigma}_1}, \ldots, \overline{\boldsymbol{\sigma}_4}$ in the stress space $(\sigma_1, \sigma_2, \sigma_3, \sigma_6)$. In addition, the sequence of rotations employed to define the orientation of the eigentensors is to a certain extent arbitrary. Then, the initial rotation could be about any of the four axes, whereas, in the subsequent five rotations, the only limitation is that no two successive rotations may be about the same axis, namely, no two successive rotations may be taken on the same plane. Hence, a number of different conventions is allowable in defining the six eigenangles as independent parameters specifying the orientation of eigentensors in the stress space $(\sigma_1, \sigma_2, \sigma_3, \sigma_6)$. However, this space is four dimensional and, as such, eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_4$ cannot be visualized. In spite of that, it is always feasible to restrict our attention to three-dimensional pictures of the four-dimensional stress space. Then, it is easily observed by projecting the stress eigentensors on an arbitrary stress space $(\sigma_i, \sigma_j, \sigma_k)$, with $i, j, k = 1, \ldots, 4$ and $i \neq j \neq k \neq i$, that vectors $\mathbf{e}_1$ to $\mathbf{e}_4$ are nonvanishing. Yet, vectors $\mathbf{e}_1$ to $\mathbf{e}_4$ are linearly dependent and, hence, in order to acquire the three eigenvectors $\mathbf{e}_m$, $m = 1, 2, 3$, corresponding to the $(\sigma_i, \sigma_j, \sigma_k)$ reference system, one has to consider the transformation of this system by means of three separate rotations through angles θ_1, θ_2 and θ_3, expressed in matrix form $\mathbf{A}_m$, $m = 1, 2, 3$, as follows:

$$\mathbf{A}_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55a}$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \tag{55b}$$

$$\mathbf{A}_3 = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{55c}$$

Accordingly, the product matrix $\mathbf{A}$ is expressed by:

$$\mathbf{A} = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1 = \begin{bmatrix} \cos\theta_1\cos\theta_3 - \sin\theta_1\cos\theta_2\sin\theta_3 & \sin\theta_1\cos\theta_3 + \cos\theta_1\cos\theta_2\sin\theta_3 & \sin\theta_2\sin\theta_3 \\ -\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_2\cos\theta_3 & \sin\theta_2\cos\theta_3 \\ \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 & \cos\theta_2 \end{bmatrix}, \tag{56}$$

which is an orthogonal matrix. Moreover, angles θ_1, θ_2 and θ_3 are known as the Euler angles, which are introduced into relations (55) to express in generalized coordinates the elements of the orthogonal transformation matrix **A**. Therefore, the projections of stress eigentensors $\overline{\boldsymbol{\sigma}_i}$, $\overline{\boldsymbol{\sigma}_j}$ and $\overline{\boldsymbol{\sigma}_k}$ are represented by a set of three orthogonal vectors with associated unit vectors $\mathbf{e}_i$, $\mathbf{e}_j$ and $\mathbf{e}_k$ having as direction cosines:

$$\mathbf{e}_i = [\sin\theta_1 \sin\theta_2, -\cos\theta_1 \sin\theta_2, \cos\theta_2]^T \tag{57a}$$

$$\mathbf{e}_j = [-\sin\theta_1 \cos\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3, \cos\theta_1 \cos\theta_2 \cos\theta_3 - \sin\theta_1 \sin\theta_3, \sin\theta_2 \cos\theta_3]^T \tag{57b}$$

$$\mathbf{e}_k = [-\sin\theta_1 \cos\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_3, \cos\theta_1 \cos\theta_2 \sin\theta_3 + \sin\theta_1 \cos\theta_3, \sin\theta_2 \sin\theta_3]^T. \tag{57c}$$

It is possible to carry out the transformation from a given Cartesian coordinate system to another by means of three successive angular displacements θ_1, θ_2, θ_3, performed in a specific sequence. Initially, frame $O\sigma_i\sigma_j\sigma_k$ is rotated through an angle θ_1 counterclockwise with respect to the $O\sigma_k \equiv O\Phi''$ axis. The resulting $O\Phi\Phi'\Phi''$ coordinate system is then rotated by an angle θ_2 about the $O\Phi \equiv O\Theta''$ axis, thus forming the subsequent system $O\Theta\Theta'\Theta''$, which is finally rotated about the $O\Omega$ axis by an angle θ_3, hence producing the final frame $O\overline{\boldsymbol{\sigma}_1}\overline{\boldsymbol{\sigma}_2}\overline{\boldsymbol{\sigma}_3} \equiv O\Omega\Omega'\Omega''$. Therefore, as seen in Fig. 2, the unit vectors $\mathbf{e}_j$ and $\mathbf{e}_k$ lie on plane $O\Theta'\Theta''$, subtending with plane $O\sigma_k\Theta''$ an angle equal to $(\pi/2 - \theta_2)$. In addition, the Θ'' axis is inclined to the σ_j axis by an angle $(\pi/2 - \theta_1)$, and the $\mathbf{e}_j$ and $\mathbf{e}_k$ unit vectors subtend an angle θ_3 with axes Θ' and Θ''.

6. Bounds of the components of the compliance tensor

It is generally accepted within the domain of classical elasticity that the existence of the thermodynamical constraint of positive-definite elastic potential sets restrictive bounds on the values of the components of the compliance tensor **S**. These constraints entailed on the elements of the general anisotropic compliance matrix were established by Voigt (1910), whereas, since then, they have been proclaimed by Born & Huang (1954) as well as by Hearmon (1961). Considering now the conditions imposed on the elastic constants of isotropic media, these are all well known and found in Love (1927). Furthermore, the restrictions applicable to media belonging to the cubic or hexagonal crystal systems are explicitly stated by Nye (1957).

Relations for the bounds of elastic compliances for transversely isotropic media were determined independently by Eubanks & Sternberg (1954), as well as by Lempriere (1968) and Christensen (1979), employing mathematically equivalent formulations, which guaranteed positive values for the elastic potential. Lempriere (1968) also examined the restrictions on the components of the compliance tensor **S** valid for orthotropic media. Recently, the bounds for the values of the elastic compliances were successfully obtained by following an alternative method based on the spectral decomposition

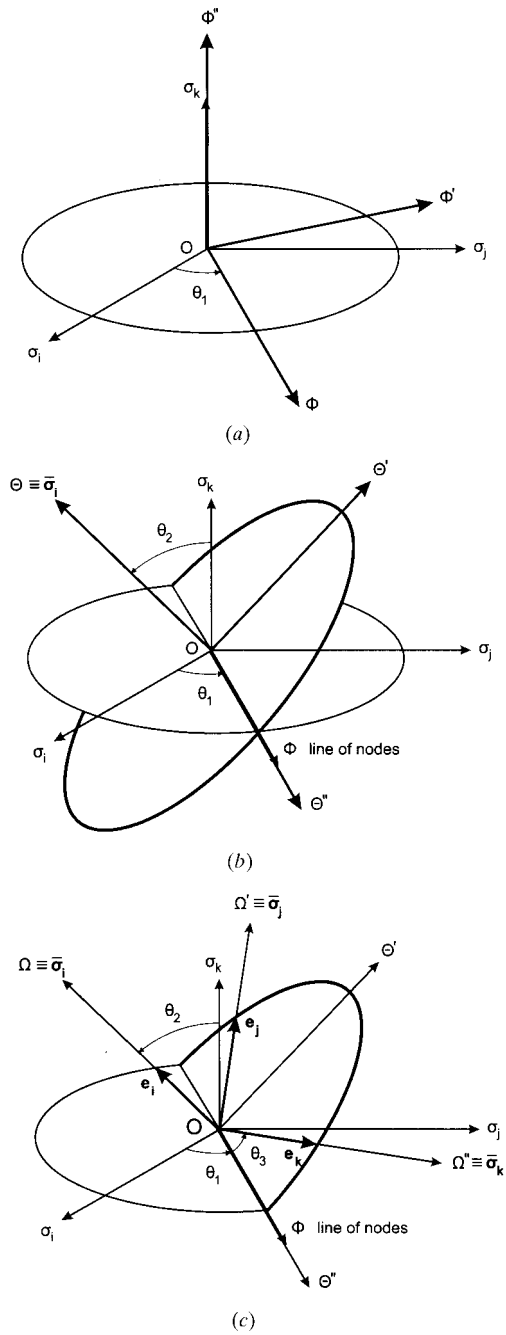

Fig. 2. (*a*), (*b*) The rotations defining the Euler angles θ_1, θ_2 and θ_3, and (*c*) geometric representation of the stress eigentensors of the monoclinic compliance fourth-rank tensor **S** in the $(\sigma_i, \sigma_j, \sigma_k)$ stress frame.

Table 1. *The values of the elastic compliances (in units of 10^{-2} GPa^{-1}) for a series of crystalline media belonging to the monoclinic system*

Crystals of the monoclinic system

		Elastic compliances ($\times 10^{-2}$ GPa^{-1})												
Symbol	Material	s_{11}	s_{22}	s_{33}	s_{44}	s_{55}	s_{66}	s_{12}	s_{13}	s_{23}	s_{16}	s_{26}	s_{36}	s_{45}
$K_2(C_4H_4O_6)\cdot 0.5H_2O$	Dipotassium tartrate (DKT1)	4.75	3.53	2.40	11.4	10.2	12.3	−1.74	−0.80	−0.62	−0.75	0.80	−1.40	−0.68
$K_2(C_4H_4O_6)\cdot 0.5H_2O$	Dipotassium tartrate (DKT2)	3.87	3.37	2.26	10.4	8.2	11.9	−1.06	−1.64	−0.07	0.85	−0.54	−0.65	0.55
$C_2H_6N_2\cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT1)	3.34	3.65	10.0	19.2	11.7	19.1	−0.3	−3.0	−1.8	−1.7	1.5	−2.65	0.38
$C_2H_6N_2\cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT2)	3.9	3.6	9.8	18.7	17.2	17.4	0.2	−5.2	−1.8	−0.5	0.2	−2.5	−0.2
$Na_2S_2O_3$	Sodium thiosulfate	5.02	15.6	6.74	22.3	32.7	21.2	−3.23	−0.62	−7.19	1.52	−18.2	11.0	10.0

analysis and the application of this analysis to both transversely isotropic media (Theocaris & Philippidis, 1991) and plates (Theocaris & Sokolis, 1998).

One of the very interesting features of the spectral analysis is its simplicity and clarity in proving the positive-definite character of the elastic strain energy. Given relations (49) and (50), it is immediately noted that, in order for the total elastic strain-energy density to be positive definite, the eigenvalues of the compliance tensor **S** need be positive definite:

$$\lambda_m > 0, \quad m = \{1, \ldots, 6\}. \tag{58}$$

This constraint requires that

$$\{s_{11}, s_{22}, s_{33}, s_{44}, s_{55}, s_{66}\} > 0 \tag{59a}$$

$$s_{12}^2 < s_{11}s_{22}, \quad s_{13}^2 < s_{11}s_{33}, \quad s_{23}^2 < s_{22}s_{33} \tag{59b}$$

$$s_{16}^2 < s_{11}s_{66}, \quad s_{26}^2 < s_{22}s_{66}, \quad s_{36}^2 < s_{33}s_{66}, \quad s_{45}^2 < s_{44}s_{55} \tag{59c}$$

$$s_{11}(s_{22}s_{33} - s_{23}^2) - s_{12}(s_{12}s_{33} - s_{13}s_{23}) + s_{13}(s_{12}s_{23} - s_{13}s_{22}) > 0 \tag{59d}$$

$$s_{11}(s_{22}s_{66} - s_{26}^2) - s_{12}(s_{12}s_{66} - s_{16}s_{26}) + s_{16}(s_{12}s_{26} - s_{16}s_{22}) > 0 \tag{59e}$$

$$s_{11}(s_{33}s_{66} - s_{36}^2) - s_{13}(s_{13}s_{66} - s_{16}s_{36}) + s_{16}(s_{13}s_{36} - s_{16}s_{33}) > 0 \tag{59f}$$

$$s_{22}(s_{33}s_{66} - s_{36}^2) - s_{23}(s_{23}s_{66} - s_{26}s_{36}) + s_{26}(s_{23}s_{36} - s_{26}s_{33}) > 0 \tag{59g}$$

$$\begin{aligned} &s_{16}s_{26}(s_{33}s_{12} - s_{13}s_{23}) + s_{26}s_{36}(s_{11}s_{23} - s_{12}s_{13}) \\ &+ s_{16}s_{36}(s_{22}s_{13} - s_{12}s_{23}) + \frac{s_{66}}{2}(s_{11}s_{22}s_{33} + 2s_{12}s_{13}s_{23} \\ &- s_{11}s_{23}^2 - s_{22}s_{13}^2 - s_{33}s_{12}^2) + \frac{s_{16}^2}{2}(s_{23}^2 - s_{22}s_{33}) \\ &+ \frac{s_{26}^2}{2}(s_{13}^2 - s_{11}s_{33}) + \frac{s_{36}^2}{2}(s_{12}^2 - s_{11}s_{22}) > 0. \end{aligned} \tag{59h}$$

It is essential that inequalities (59) are all simultaneously satisfied in order for the elastic strain-energy density to be positive definite. Hence, bounds of the elastic constants based on partial fulfilment of these inequalities are considered improper and should be excluded.

7. Numerical examples

The experimentally measured values of elastic compliance-tensor components for several common representative inorganic crystals belonging to the monoclinic system are listed in Table 1 (Landolt-Bornstein, 1979, 1984). It must be pointed out that the multiple entries appearing in Table 1 for dipotassium tartrate (DKT) and for ethylenediamine tartrate (EDT) are due to substantial disagreement between different investigators using usually reliable techniques.

Now, in order to fix ideas, we shall try to evaluate, using the numerical values of the compliance components for dipotassium tartrate (DKT1), the eigenvalues, the eigenangles and the stress and strain eigenvectors. The experimental values, in units of 10^{-2} GPa^{-1}, are as follows:

$$s_{11} = 4.75,\ s_{22} = 3.53,\ s_{33} = 2.40,\ s_{44} = 11.4 \tag{60a}$$

$$s_{55} = 10.2,\ s_{66} = 12.3,\ s_{12} = -1.74,\ s_{13} = -0.80 \tag{60b}$$

$$s_{23} = -0.62,\ s_{16} = -0.75,\ s_{26} = 0.80,\ s_{36} = -1.40,\ s_{45} = -0.68. \tag{60c}$$

For this dipotassium tartrate (DKT1), the eigenvalues, in units of TPa^{-1}, defined by relations (26), are

$$\lambda_1 = 30.575, \quad \lambda_2 = 12.857 \tag{61a}$$

$$\lambda_3 = 69.183, \quad \lambda_4 = 55.685 \tag{61b}$$

$$\lambda_5 = 58.534, \quad \lambda_6 = 49.466. \tag{61c}$$

The eigenvalues $\lambda_1, \ldots, \lambda_6$ of the compliance tensor **S**, in units of TPa^{-1}, for the remaining inorganic crystals belonging to the monoclinic system are tabulated in Table 2. Moreover, the eigenangles $\rho, \nu, \mu, \psi, \theta, \omega$ and φ, defined by relations (33), are evaluated to be

Table 2. *The values of the six eigenvalues (in units of TPa^{-1}) of the compliance fourth-rank tensor* ***S*** *for a series of crystalline media belonging to the monoclinic system*

Crystals of the monoclinic system		Eigenvalues (TPa^{-1})					
Symbol	Material	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
$K_2(C_4H_4O_6) \cdot 0.5H_2O$	Dipotassium tartrate (DKT1)	30.575	12.857	69.183	55.685	58.534	49.466
$K_2(C_4H_4O_6) \cdot 0.5H_2O$	Dipotassium tartrate (DKT1)	31.454	10.532	65.249	47.265	52.649	40.351
$C_2H_6N_2 \cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT1)	34.748	14.846	124.77	91.299	96.096	58.404
$C_2H_6N_2 \cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT2)	35.004	50.395	135.15	84.809	93.631	85.869
$Na_2S_2O_3$	Sodium thiosulfate	13.716	−6.678	310.69	61.864	193.86	81.144

Table 3. *The values of the set of eigenangles (°) of the compliance fourth-rank tensor* ***S*** *for a series of crystalline media belonging to the monoclinic system*

Crystals of the monoclinic system		Eigenangles (°)						
Symbol	Material	ρ	ν	μ	θ	ω	φ	ψ
$K_2(C_4H_4O_6) \cdot 0.5H_2O$	Dipotassium tartrate (DKT1)	15.89	22.64	−52.72	129.79	168.46	148.32	24.29
$K_2(C_4H_4O_6) \cdot 0.5H_2O$	Dipotassium tartrate (DKT2)	−19.47	−21.58	49.73	−60.57	−17.70	170.90	166.72
$C_2H_6N_2 \cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT1)	26.26	−2.03	−153.78	88.96	63.24	154.61	2.89
$C_2H_6N_2 \cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT2)	−9.61	2.61	15.58	89.02	122.10	17.16	172.53
$Na_2S_2O_3$	Sodium thiosulfate	27.67	16.99	260.49	36.30	131.51	174.41	58.73

$$\psi = 24.299^\circ, \quad \rho = 15.892^\circ,$$
$$\nu = 22.641^\circ, \quad \mu = -52.723^\circ \tag{62a}$$
$$\theta = 129.792^\circ, \quad \omega = 168.460^\circ, \quad \varphi = 148.321^\circ \tag{62b}$$

and the eigenangles of the compliance tensor **S** for the remaining representative monoclinic crystals are given in Table 3.

Then, the eigenvectors of the compliance fourth-rank tensor **S**, as defined in (51) and (52), are found to be given by

$$\mathbf{e}_1 = [0.315, 0.671, -0.616, 0, 0 - 0.266]^T \tag{63a}$$
$$\mathbf{e}_2 = [0.444, 0.510, 0.724, 0, 0, 0.137]^T \tag{63b}$$
$$\mathbf{e}_3 = [0.452, -0.390, 0.146, 0, 0, -0.788]^T \tag{63c}$$
$$\mathbf{e}_4 = [0.706, -0.370, -0.274, 0, 0, 0.538]^T \tag{63d}$$
$$\mathbf{e}_5 = [0, 0, 0, 0.912, -0.411, 0]^T \tag{63e}$$
$$\mathbf{e}_6 = [0, 0, 0, -0.411, -0.912, 0]^T. \tag{63f}$$

Table 4 presents the eigenvectors of the remaining monoclinic crystals.

8. Discussion

The most important property of spectral analysis is its ability to expose in a very natural way the analogy in the elastic characteristics of isotropic and anisotropic media. For instance, the form of the compliance tensor **S** was established for an isotropic body during the first quarter of the last century:

$$\mathbf{S} = \frac{1}{\lambda}(\mathbf{1} \otimes \mathbf{1}) + \frac{1}{2\mu}\mathbf{I} \tag{64}$$

in terms of Lamé elastic moduli λ and μ. Indeed, the analogy between this expression and the spectral expansion (36) of the compliance tensor **S** for anisotropic media belonging to the monoclinic system is easily recognized, whereas instead of Lamé moduli one has the eigenvalues of the compliance tensor **S**.

Furthermore, it is well known that the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ tensors of linearly isotropic elastic media are decomposed into deviatoric and hydrostatic parts:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_D + \tfrac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_D + \tfrac{1}{3}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1}, \tag{65}$$

in which subscript D in $\boldsymbol{\sigma}_D$ and $\boldsymbol{\varepsilon}_D$ denotes the deviatoric parts and the second terms denote the hydrostatic parts $\boldsymbol{\sigma}_P$ and $\boldsymbol{\varepsilon}_P$ of the stress and strain tensors, respectively. Then, this characteristic of isotropic elasticity encounters its analogy in the spectral decomposition of the stress and strain tensors into six distinct non-interacting stress states. However, it should be made clear that, whereas the deviatoric and hydrostatic eigentensors remain constant for all isotropic materials, the corresponding eigentensors of anisotropic elasticity are dependent on the elastic compliance components, thus obtaining different values for different media.

Besides, the decomposition of the stress and strain tensors in isotropic elastic bodies into hydrostatic and deviatoric constituents results in an equivalent decomposition of Hooke's law for isotropic materials into two equations:

Table 4. *The components of the set of eigenvectors of the compliance fourth-rank tensor* ***S*** *for a series of crystalline media belonging to the monoclinic system*

Crystals of the monoclinic system			
Symbol	Material	Eigenvectors	
$K_2(C_4H_4O_6) \cdot 0.5H_2O$	Dipotassium tartrate (DKT1)	$\mathbf{e}_1 = [0.315, 0.671, -0.616, 0, 0, -0.266]^T$	$\mathbf{e}_2 = [0.444, 0.510, 0.724, 0, 0, 0.137]^T$
		$\mathbf{e}_3 = [0.452, -0.390, 0.146, 0, 0, -0.788]^T$	$\mathbf{e}_4 = [0.706, -0.370, -0.274, 0, 0, 0.538]^T$
		$\mathbf{e}_5 = [0, 0, 0, 0.912, -0.411, 0]^T$	$\mathbf{e}_6 = [0, 0, 0, -0.411, -0.912, 0]^T$
$K_2(C_4H_4O_6) \cdot 0.5H_2O$	Dipotassium tartrate (DKT2)	$\mathbf{e}_1 = [0.215, 0.860, -0.463, 0, 0, -0.005]^T$	$\mathbf{e}_2 = [-0.556, -0.282, -0.782, 0, 0, -0.027]^T$
		$\mathbf{e}_3 = [-0.441, 0.243, 0.256, 0, 0, -0.826]^T$	$\mathbf{e}_4 = [-0.670, 0.347, 0.333, 0, 0, 0.566]^T$
		$\mathbf{e}_5 = [0, 0, 0, -0.973, -0.230, 0]^T$	$\mathbf{e}_6 = [0, 0, 0, 0.230, -0.973, 0]^T$
$C_2H_6N_2 \cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT1)	$\mathbf{e}_1 = [-0.366, 0.903, 0.016, 0, 0, -0.225]^T$	$\mathbf{e}_2 = [0.821, 0.367, 0.404, 0, 0, 0.168]^T$
		$\mathbf{e}_3 = [-0.188, -0.221, 0.801, 0, 0, -0.525]^T$	$\mathbf{e}_4 = [0.396, 0.032, -0.443, 0, 0, -0.804]^T$
		$\mathbf{e}_5 = [0, 0, 0, 0.999, 0.051, 0]^T$	$\mathbf{e}_6 = [0, 0, 0, 0.051, -0.999, 0]^T$
$C_2H_6N_2 \cdot C_4H_6O_6$	Ethylenediamine tartrate (EDT2)	$\mathbf{e}_1 = [0.295, -0.955, -0.017, 0, 0, 0.041]^T$	$\mathbf{e}_2 = [-0.803, -0.246, -0.524, 0, 0, -0.143]^T$
		$\mathbf{e}_3 = [-0.445, -0.165, 0.835, 0, 0, -0.279]^T$	$\mathbf{e}_4 = [-0.265, -0.045, 0.167, 0, 0, 0.949]^T$
		$\mathbf{e}_5 = [0, 0, 0, -0.992, 0.130, 0]^T$	$\mathbf{e}_6 = [0, 0, 0, -0.130, -0.992, 0]^T$
$Na_2S_2O_3$	Sodium thiosulfate	$\mathbf{e}_1 = [0.532, 0.454, 0.714, 0, 0, -0.030]^T$	$\mathbf{e}_2 = [-0.092, -0.492, 0.347, 0, 0, -0.793]^T$
		$\mathbf{e}_3 = [-0.101, 0.696, -0.393, 0, 0, -0.592]^T$	$\mathbf{e}_4 = [0.835, -0.259, -0.464, 0, 0, -0.140]^T$
		$\mathbf{e}_5 = [0, 0, 0, 0.519, 0.855, 0]^T$	$\mathbf{e}_6 = [0, 0, 0, -0.855, 0.519, 0]^T$

$$\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1} = \tfrac{1}{3}(3\lambda + 2\mu)(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{1}, \quad \boldsymbol{\sigma}_D = 2\mu\boldsymbol{\varepsilon}_D \tag{66}$$

between the hydrostatic and deviatoric stress and strain eigentensors. Then, the generalized anisotropic Hooke's law valid for monoclinic media, which is formulated alternatively in the equivalent form of a system of six non-interacting mutually orthogonal laws of direct proportionality, expressed by relations (47), may be thought of as a generalization of equations (66) above.

Finally, the stress and strain eigentensors were proven to partition directly the elastic strain-energy density into distinct strain-energy constituents. Again, an analogy is revealed between the splitting of the total elastic strain-energy density of the monoclinic medium and the corresponding splitting valid for the isotropic medium, which is given in the form

$$\begin{aligned} T(\boldsymbol{\sigma}) &= T(\boldsymbol{\sigma}_P) + T(\boldsymbol{\sigma}_D) \\ &= \frac{1}{18K}(\operatorname{tr}\boldsymbol{\sigma})^2 + \frac{1}{2G}\left[\operatorname{tr}\boldsymbol{\sigma}^2 - \tfrac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})^2\right], \end{aligned} \tag{67}$$

where $T(\boldsymbol{\sigma}_D)$ is the deviatoric strain energy and $T(\boldsymbol{\sigma}_P)$ is the hydrostatic strain energy, corresponding to the deviatoric and hydrostatic stresses and strains respectively.

However, the decomposition of the elastic potential that is valid for the isotropic medium is not valid for the monoclinic one. Hence, it is shown that a generalization of the decomposition of the elastic strain-energy density into components corresponding to sole dilatational and distortional types of energy, valid for the isotropic medium as well as for cubic crystals, is impossible for the monoclinic medium, since the second-rank eigentensors of the compliance fourth-rank tensor **S** do not include the spherical tensor **1**. Thus, an explanation is given for the failure of the studies undertaken (Olszak & Urbanowski, 1956; Olszak & Ostrowska-Maciejewska, 1985), which aimed to generalize the Huber–Mises–Hencky criterion to hold for anisotropic media, therefore establishing the distortional component of the elastic strain-energy density as the critical failure quantity.

In conclusion, the spectral decomposition of the compliance fourth-rank tensor **S** allows the possibility of generalization of well known characteristics of isotropic linear elastic bodies to anisotropic ones, thus offering to the theory of anisotropic media in the elastic domain a status comparable to that of isotropic elasticity.

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